Graham's number

Graham's number is an enormous number that arises as an upper bound on the answer of a problem in the mathematical field of Ramsey theory. It is named after mathematician Ronald Graham, who used the number as a simplified explanation of the upper bounds of the problem he was working on in conversations with popular science writer Martin Gardner. Gardner later described the number in *Scientific American* in 1977, introducing it to the general public. At the time of its introduction, it was the largest specific positive integer ever to have been used in a published mathematical proof. The number was published in the 1980 *Guinness Book of World Records*, aiding to its popular interest. Other specific integers (such as TREE(3)) known to be far larger than Graham's number have since appeared in many serious mathematical proofs, for example in connection with Harvey Friedman's various finite forms of Kruskal's theorem. Additionally, smaller upper bounds on the Ramsey theory problem from which Graham's number derived have since been proven to be valid.

Graham's number is much larger than many other large numbers such as Skewes' number and Moser's number, both of which are in turn much larger than a googolplex. As with these, it is so large that the observable universe is far too small to contain an ordinary digital representation of Graham's number, assuming that each digit occupies one Planck volume, possibly the smallest measurable space. But even the number of digits in this digital representation of Graham's number would itself be a number larger than its digital representation cannot be represented in the observable universe. Nor even can the number of digits of *that* number—and so forth, for a number of times far exceeding the total number of Planck volumes in the observable universe. Thus Graham's number cannot even be expressed in this way by power towers of the form $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$.

However, Graham's number can be explicitly given by computable recursive formulas using Knuth's up-arrow notation or equivalent, as was done by Graham. As there is a recursive formula to define it, it is much smaller than typical busy beaver numbers. Though too large to be computed in full, the sequence of digits of Graham's number can be computed explicitly through simple algorithms. The last 12 digits are 2464464195387. With Knuth's up-arrow notation, Graham's number is g_{64}, where

$$g_n = \begin{cases} 3 \uparrow\uparrow\uparrow\uparrow 3, & n = 1 \\ 3 \uparrow^{g_{n-1}} 3, & n \geq 2, n \in \mathbb{N} \end{cases}$$

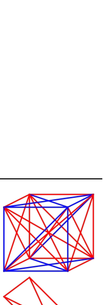
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Context

Graham's number is connected to the following problem in Ramsey theory:

> Connect each pair of geometric vertices of an *n*-dimensional hypercube to obtain a complete graph on 2^n vertices. Colour each of the edges of this graph either red or blue. What is the smallest value of *n* for which *every* such colouring contains at least one single-coloured complete subgraph on four coplanar vertices?



Example of a 2-colored 3-dimensional cube containing one single-coloured 4-vertex coplanar complete subgraph. The subgraph is shown below the cube. Note that this cube would contain no such subgraph if, for example, the bottom edge in the present subgraph were replaced by a blue edge – thus proving by counterexample that $N^* > 3$.

In 1971, Graham and Rothschild proved that this problem has a solution N^*, giving as a bound $6 \leq N^* \leq N$, with N being a large but explicitly defined number $F^7(12) = F(F(F(F(F(F(F(12)))))))$, where $F(n) = 2\uparrow^n 3$ in Knuth's up-arrow notation; the number is between $4 \to 2 \to 8 \to 2$ and $2 \to 3 \to 9 \to 2$ in Conway chained arrow notation.[1] This was reduced in 2014 via upper bounds on the Hales–Jewett number to $N^* = 2\uparrow\uparrow\uparrow 6$.[2] The lower bound of 6 was later improved to 11 by Geoffrey Exoo in 2003,[3] and to 13 by Jerome Barkley in 2008.[4] Thus, the best known bounds for N^* are $13 \leq N^* \leq N$.

Graham's number, G, is much larger than N: $f^{64}(4)$, where $f(n) = 3\uparrow^n 3$. This weaker upper bound for the problem, attributed to Graham, was eventually published and named by Martin Gardner in *Scientific American* in November 1977.[5]

Publication

The number gained a degree of popular attention when Martin Gardner described it in the "Mathematical Games" section of *Scientific American* in November 1977, writing that Graham had recently established, in an unpublished proof, "a bound so vast that it holds the record for the largest number ever used in a serious mathematical proof." The 1980 *Guinness Book of World Records* repeated Gardner's claim, adding to the popular interest in this number. According to physicist John Baez, Graham invented the quantity now known as Graham's number in conversation with Gardner. While Graham was trying to explain a result in Ramsey theory which he had derived with his collaborator Bruce Lee Rothschild, Graham found that the quantity now known as Graham's number was easier to explain than the actual number appearing in the proof. Because the number which Graham described to Gardner is larger than the number in the paper itself, both are valid upper bounds for the solution to the problem studied by Graham and Rothschild.[6]

Definition

Using Knuth's up-arrow notation, Graham's number G (as defined in Gardner's *Scientific American* article) is

$$G = \underbrace{3\uparrow\uparrow\cdots\cdots\cdots\cdots\uparrow 3}_{\begin{matrix}3\uparrow\uparrow\cdots\cdots\cdots\uparrow 3\\ \vdots \\ 3\uparrow\uparrow\cdots\uparrow 3\\ 3\uparrow\uparrow\uparrow 3\end{matrix}} \text{64 layers}$$

where the number of $\uparrow$s in each subsequent layer is specified by the value of the next layer below it; that is,

$$G = g_{64},$$

where $g_1 = 3\uparrow\uparrow\uparrow\uparrow 3$, $g_n = 3\uparrow^{g_{n-1}} 3$,

and where a superscript on an up-arrow indicates how many arrows there are. In other words, G is calculated in 64 steps: the first step is to calculate g_1 with four up-arrows between 3s; the second step is to calculate g_2 with g_1 up-arrows between 3s; the third step is to calculate g_3 with g_2 up-arrows between 3s; and so on, until finally calculating $G = g_{64}$ with g_{63} up-arrows between 3s.

Equivalently,

$$G = f^{64}(4), \text{ where } f(n) = 3\uparrow^n 3,$$

and the superscript on f indicates an iteration of the function, e.g., $f^4(n) = f(f(f(f(n))))$. Expressed in terms of the family of hyperoperations $\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2, \cdots$, the function f is the particular sequence $f(n) = \mathbf{H}_{n+3}(\mathbf{3}, \mathbf{3})$, which is a version of the rapidly growing Ackermann function $A(n, n)$. (In fact, $f(n) > A(n, n)$ for all n.) The function f can also be expressed in Conway chained arrow notation as $f(n) = 3 \to 3 \to \mathbf{n}$, and this notation also provides the following bounds on G:

$$3 \to 3 \to 64 \to 2 < G < 3 \to 3 \to 65 \to 2.$$

Magnitude

To convey the difficulty of appreciating the enormous size of Graham's number, it may be helpful to express—in terms of exponentiation alone—just the first term (g_1) of the rapidly growing 64-term sequence. First, in terms of tetration ($\uparrow\uparrow\uparrow$) alone:

$$g_1 = 3\uparrow\uparrow\uparrow\uparrow 3 = 3\uparrow\uparrow\uparrow(3\uparrow\uparrow\uparrow 3) = 3\uparrow\uparrow(3\uparrow\uparrow(3\uparrow\uparrow\ \ \ldots\ (3\uparrow\uparrow 3)\ldots))$$

where the number of 3s in the expression on the right is

$$3\uparrow\uparrow\uparrow 3 = 3\uparrow\uparrow(3\uparrow\uparrow 3).$$

Now each tetration ($\uparrow\uparrow$) operation reduces to a power tower ($\uparrow$) according to the definition

$$3\uparrow\uparrow X = 3\uparrow(3\uparrow(3\uparrow\ldots(3\uparrow 3)\ldots)) = 3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}\Big\}X$$

where there are X 3s.

Thus,

$$g_1 = 3\uparrow\uparrow(3\uparrow\uparrow(3\uparrow\uparrow\ \ \ldots\ (3\uparrow\uparrow 3)\ldots)) \quad \text{where the number of 3s is} \quad 3\uparrow\uparrow(3\uparrow\uparrow 3)$$

becomes, solely in terms of repeated "exponentiation towers",

$$g_1 = 3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}\Big\}3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}\Big\}\cdots\Big\}3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}\Big\}3 \quad \text{where the number of towers is} \quad 3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}\Big\}3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}\Big\}3$$

and where the number of 3s in each tower, starting from the leftmost tower, is specified by the value of the next tower to the right.

In other words, g_1 is computed by first calculating the number of towers, $\mathbf{n} = 3\uparrow\uparrow\uparrow 3 = 3\uparrow\uparrow(3\uparrow\uparrow 3) = 7625597484987$), and then computing the n^{th} tower in the following sequence:

```
1st tower:  3
2nd tower:  3↑3↑3 (number of 3s is 3) = 7625597484987
3rd tower:  3↑3↑3↑3...↑3 (number of 3s is 7625597484987) = …
    ⋮
g1 = nth tower:  3↑3↑3↑3↑3↑3↑...↑3 (number of 3s is given by the n=(3↑↑3)th tower)
```

where the number of 3s in each successive tower is given by the tower just before it. Note that the result of calculating the third tower is the value of n, the number of towers for g_1.

The magnitude of this first term, g_1, is so large that it is practically incomprehensible, even though the above display is relatively easy to comprehend. Even n, the mere *number of towers* in this formula for g_1, is far greater than the number of Planck volumes (roughly 10^{185} of them) into which one can imagine subdividing the observable universe. And after this first term, still another 63 terms remain in the rapidly growing g sequence before Graham's number $G = g_{64}$ is reached. To illustrate just how fast this sequence grows, while g_1 is equal to $3\uparrow\uparrow\uparrow\uparrow 3$ with only four up arrows, the number of up arrows in g_2 is this incomprehensibly large number g_1.

Rightmost decimal digits

Graham's number is a "power tower" of the form $3\uparrow\uparrow n$ (with a very large value of n), so its rightmost decimal digits must satisfy certain properties common to all such towers. One of those properties is that *all such towers of height greater than d (say), have the same sequence of d rightmost decimal digits*. This is a special case of a more general property: The d rightmost decimal digits of all such towers of height greater than $d+2$, are *independent* of the topmost "3" in the tower; i.e., the topmost "3" can be changed to any other non-negative integer without affecting the d rightmost digits.

The following table illustrates, for a few values of d, how this happens. For a given height of tower and number of digits d, the full range of d-digit numbers (10^d of them) does not occur; instead, a certain smaller subset of values repeats itself in a cycle. The length of the cycle and some of the values (in parentheses) are shown in each cell of this table:

Number of different possible values of $3\uparrow 3\uparrow\ldots 3\uparrow x$ when all but the rightmost d decimal digits are ignored

Number of digits (d)	$3\uparrow x$	$3\uparrow 3\uparrow x$	$3\uparrow 3\uparrow 3\uparrow x$	$3\uparrow 3\uparrow 3\uparrow 3\uparrow x$	$3\uparrow 3\uparrow 3\uparrow 3\uparrow 3\uparrow x$
1	4 (1,3,9,7)	2 (3,7)	1 (7)	1 (7)	1 (7)
2	20 (01,03,…,87,…,67)	4 (03,27,83,**87**)	2 (27,**87**)	1 (**87**)	1 (**87**)
3	100 (001,003,…,**387**,…,667)	20 (003,027,…,**387**,…,587)	4 (027,987,227,**387**)	2 (987,**387**)	1 (**387**)

The particular rightmost d digits that are ultimately shared by all sufficiently tall towers of 3s are in bold text, and can be seen developing as the tower height increases. For any fixed number of digits d (row in the table), the number of values possible for $3\uparrow\uparrow\ldots\uparrow x$ mod 10^d, as x ranges over all nonnegative integers, is seen to decrease steadily as the height increases, until eventually reducing the "possibility set" to a single number (colored cells) when the height exceeds $d+2$.

A simple algorithm[7] for computing these digits may be described as follows: let $x = 3$, then iterate, d times, the assignment $x = 3^x \bmod 10^d$. Except for omitting any leading os, the final value assigned to x (as a base-ten numeral) is then composed of the d rightmost decimal digits of $3\uparrow\uparrow n$, for all $n > d$. (If the final value of x has fewer than d digits, then the required number of leading os must be added.)

Let k be the numerousness of these *stable* digits, which satisfy the congruence relation $G \bmod 10^k = (G^G) \bmod 10^k$).

k=f-1, where $G(f) = 3\uparrow\uparrow f$.[8] It follows that, $g_{63} = k = g_{64}$.

The algorithm above produces the following 500 rightmost decimal digits of Graham's number (or of any tower of more than 500 3s):

```
…62425956095984738395657479136551935178833455358625218
43003540126926771622672160419810652263169355188780
38814483140652526168780595552646805187137080997982
91249544378887496862882911725065008138362293491608
25493461494578871474278325082924216209818258967515568
43086938016892949892660995518169055919951195827887
17838878183462364744588222216371228818132974589
27344594584343980991996028255257518332898446158
94404240368183585156251979619986189939679854964388
02222348729678184851864390591845756728264619538?
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References

1. "Graham's number records" (http://literor.org/big/Source/Graham-Gardner/GrahamsNumber.html). literor.org. Retrieved 2014-04-09.
2. Lavrov, Mikhail; Lee, Mitchell; Mackey, John (2014). "Improved upper and lower bounds on a geometric Ramsey problem". *European Journal of Combinatorics*. **42**: 135–144. doi:10.1016/j.ejc.2014.06.003 (https://doi.org/10.1016%2Fj.ejc.2014.06.003).
3. Exoo, Geoffrey (2003). "A Euclidean Ramsey Problem". *Discrete & Computational Geometry*. **29** (2): 223–227. doi:10.1007/s00454-002-0780-5 (https://doi.org/10.1007%2Fs00454-002-0780-5). Exoo refers to the Graham & Rothschild upper bound *N* by the term "Graham's number". This is not the "Graham's number" *G* published by Martin Gardner.
4. Barkley, Jerome (2008). "Improved lower bound on an Euclidean Ramsey problem". arXiv:0811.1055 (https://arxiv.org/abs/0811.1055) [math.CO (https://arxiv.org/archive/math.CO)].
5. Martin Gardner (1977) "In which joining sets of points leads into diverse (and diverting) paths" (http://literor.org/big/Source/Graham-Gardner/GrahamsNumber.html). *Scientific American*, November 1977
6. John Baez (2013). "A while back I told you about Graham's number..." (https://plus.google.com/117663015413546257905/posts/KJTpJhTZCrO)
7. "The Math Forum @ Drexel ("Last Eight Digits of 3↑↑") (http://mathforum.org/library/drmath/view/51625.html). Mathforum.org. Retrieved 2014-04-09.
8. Ripà, Marco (2011). *La strana coda della serie n^n^…^n (in Italian)*. UNI Service. ISBN 978-88-6178-789-6.

Bibliography

- Gardner, Martin (November 1977). "Mathematical Games" (http://www.nature.com/scientificamerican/journal/v237/n5/pdf/scientificamerican1177-18.pdf) (PDF). *Scientific American*. **237**: 18–28. Bibcode:1977SciAm.237e..18G (http://adsabs.harvard.edu/abs/1977SciAm.237e..18G). doi:10.1038/scientificamerican1177-18 (https://doi.org/10.1038%2Fscientificamerican1177-18). ; reprinted (revised) in Gardner (2001), cited below.
- Gardner, Martin (1989). *Penrose Tiles to Trapdoor Ciphers*. Washington, D.C.: Mathematical Association of America. ISBN 0-88385-521-6.
- Gardner, Martin (2001). *The Colossal Book of Mathematics: Classic Puzzles, Paradoxes, and Problems*. New York, NY: Norton. ISBN 0-393-02023-1.
- Graham, R. L.; Rothschild, B. L. (1971). "Ramsey's Theorem for *n*-Parameter Sets" (http://www.ams.org/journals/tran/1971-159-00/S0002-9947-1971-0284352-8/S0002-9947-1971-0284352-8.pdf) (PDF). *Transactions of the American Mathematical Society*. **159**: 257–292. doi:10.2307/1996010 (https://doi.org/10.2307%2F1996010). JSTOR 1996010 (https://www.jstor.org/stable/1996010). The explicit formula for *N* appears on p. 290. This is not the "Graham's number" *G* published by Martin Gardner.
- Graham, R. L.; Rothschild, B. L. (1978). "Ramsey Theory". In Rota, G-C. *Studies in Combinatorics (MAA Studies in Mathematics)*. **17**. Mathematical Association of America. pp. 80–99. ISBN 0-88385-117-2. On p. 90, in stating "the best available estimate" for the solution, the explicit formula for *N* is repeated from the 1971 paper.

External links

- Sbiis Saibian's article on Graham's number (https://sites.google.com/site/largenumbers/home/3-2/Graham)
- "A Ramsey Problem on Hypercubes (http://isu.indstate.edu/ge/GEOMETRY/cubes.html)" by Geoff Exoo
- Mathworld article on Graham's number (http://mathworld.wolfram.com/GrahamsNumber.html)
- How to calculate Graham's number (http://www-users.cs.york.ac.uk/susan/cyc/g/graham.htm)
- Conceptualizing Graham's number (http://waitbutwhy.com/2014/11/1000000-grahams-number.html)
- Some Ramsey results for the *n*-cube (http://www.math.ucsd.edu/~fan/ron/papers/pre_cube.pdf) prepublication mentions Graham's number
- Padilla, Tony; Parker, Matt. "Graham's Number" (http://www.numberphile.com/videos/grahamsnumber.html). *Numberphile*. Brady Haran.
- Ron Graham. "What is Graham's Number? (feat Ron Graham)" (https://www.youtube.com/watch?v=HXBbihEe3nA) (video). *Numberphile*. Brady Haran.
- Ron Graham. "How Big is Graham's Number? (feat Ron Graham)" (https://www.youtube.com/watch?v=GuigptwIVNo) (video). *Numberphile*. Brady Haran.
- The last 16 million digits of Graham's number by the (http://ankokudan.org/dl/others/Graham-16M.pdf)Darkside communication group

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