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Law of total probability

In probability theory, the **law** (or **formula**) **of total probability** is a fundamental rule relating marginal probabilities to conditional probabilities. It expresses the total probability of an outcome which can be realized via several distinct events—hence the name.

Contents

Statement

The law of total probability is[1] a theorem that, in its discrete case, states if $\{B_n : n = 1, 2, 3, \ldots\}$ is a finite or countably infinite partition of a sample space (in other words, a set of pairwise disjoint events whose union is the entire sample space) and each event B_n is measurable, then for any event A of the same probability space:

$$P(A) = \sum_n P(A \cap B_n)$$

or, alternatively,[1]

$$P(A) = \sum_n P(A \mid B_n) P(B_n),$$

where, for any n for which $P(B_n) = 0$ these terms are simply omitted from the summation, because $P(A \mid B_n)$ is finite.

The summation can be interpreted as a weighted average, and consequently the marginal probability, $P(A)$, is sometimes called "average probability";[2] "overall probability" is sometimes used in less formal writings.[3]

The law of total probability, can also be stated for conditional probabilities.

$$P(A \mid C) = \sum_n P(A \mid C \cap B_n) P(B_n \mid C)$$

Taking the B_n as above, and assuming C is an event independent of any of the B_n:

$$P(A \mid C) = \sum_n P(A \mid C \cap B_n) P(B_n)$$

Informal formulation

The above mathematical statement might be interpreted as follows: *given an event A, with known conditional probabilities given any of the B_n events, each with a known probability itself, what is the total probability that A will happen?* The answer to this question is given by $P(A)$.

Continuous case

The law of total probability extends to the case of conditioning on events generated by continuous random variables. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose X is a random variable with distribution function F_X, and A an event on $(\Omega, \mathcal{F}, P)$. Then the law of total probability states

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x) dF_X(x).$$

If X admits a density function f_X, then the result is

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x) f_X(x) dx.$$

Moreover, for the specific case where $A = \{Y \in B\}$, where B is a borel set, then this yields

$$P(Y \in B) = \int_{-\infty}^{\infty} P(Y \in B|X = x) f_X(x) dx.$$

Example

Suppose that two factories supply light bulbs to the market. Factory *X*'s bulbs work for over 5000 hours in 99% of cases, whereas factory *Y*'s bulbs work for over 5000 hours in 95% of cases. It is known that factory *X* supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

Applying the law of total probability, we have:

$$\begin{aligned} P(A) &= P(A \mid B_X) \cdot P(B_X) + P(A \mid B_Y) \cdot P(B_Y) \\ &= \frac{99}{100} \cdot \frac{6}{10} + \frac{95}{100} \cdot \frac{4}{10} = \frac{594 + 380}{1000} = \frac{974}{1000} \end{aligned}$$

where

- $P(B_X) = \frac{6}{10}$ is the probability that the purchased bulb was manufactured by factory *X*;
- $P(B_Y) = \frac{4}{10}$ is the probability that the purchased bulb was manufactured by factory *Y*;
- $P(A \mid B_X) = \frac{99}{100}$ is the probability that a bulb manufactured by *X* will work for over 5000 hours;
- $P(A \mid B_Y) = \frac{95}{100}$ is the probability that a bulb manufactured by *Y* will work for over 5000 hours.

Thus each purchased light bulb has a 97.4% chance to work for more than 5000 hours.

Other names

The term ***law of total probability*** is sometimes taken to mean the **law of alternatives**, which is a special case of the law of total probability applying to discrete random variables. One author uses the terminology of the "Rule of Average Conditional Probabilities",[4] while another refers to it as the "continuous law of alternatives" in the continuous case.[5] This result is given by Grimmett and Welsh[6] as the **partition theorem**, a name that they also give to the related law of total expectation.

See also

- Law of total expectation
- Law of total variance
- Law of total covariance
- Law of total cumulance
- Marginal distribution

Notes

1. Zwillinger, D., Kokoska, S. (2000) *CRC Standard Probability and Statistics Tables and Formulae*, CRC Press. ISBN 1-58488-059-7 page 31.
2. Paul E. Pfeiffer (1978). *Concepts of probability theory* (https://books.google.com/books?id=_mayRBczVRwC&pg=PA47). Courier Dover Publications. pp. 47–48. ISBN 978-0-486-63677-1.
3. Deborah Rumsey (2006). *Probability for dummies* (https://books.google.com/books?id=Vj3NZ59ZcnoC&pg=PA58). For Dummies. p. 58. ISBN 978-0-471-75141-0.
4. Jim Pitman (1993). *Probability* (https://books.google.com/books?id=AoDkBwAAQBAJ&q=pitman%20probability&pg=PA41). Springer. p. 41. ISBN 0-387-97974-3.
5. Kenneth Baclawski (2008). *Introduction to probability with R* (https://books.google.com/books?id=Kglc9g5IPf4C&pg=PA179). CRC Press. p. 179. ISBN 978-1-4200-6521-3.
6. *Probability: An Introduction*, by Geoffrey Grimmett and Dominic Welsh, Oxford Science Publications, 1986, Theorem 1B.

References

- *Introduction to Probability and Statistics* by Robert J. Beaver, Barbara M. Beaver, Thomson Brooks/Cole, 2005, page 159.
- *Theory of Statistics*, by Mark J. Schervish, Springer, 1995.
- *Schaum's Outline of Probability, Second Edition*, by John J. Schiller, Seymour Lipschutz, McGraw–Hill Professional, 2010, page 89.
- *A First Course in Stochastic Models*, by H. C. Tijms, John Wiley and Sons, 2003, pages 431–432.
- *An Intermediate Course in Probability*, by Alan Gut, Springer, 1995, pages 5–6.